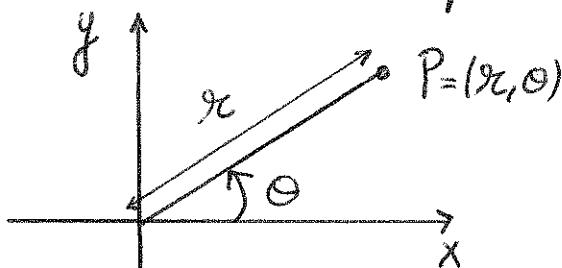


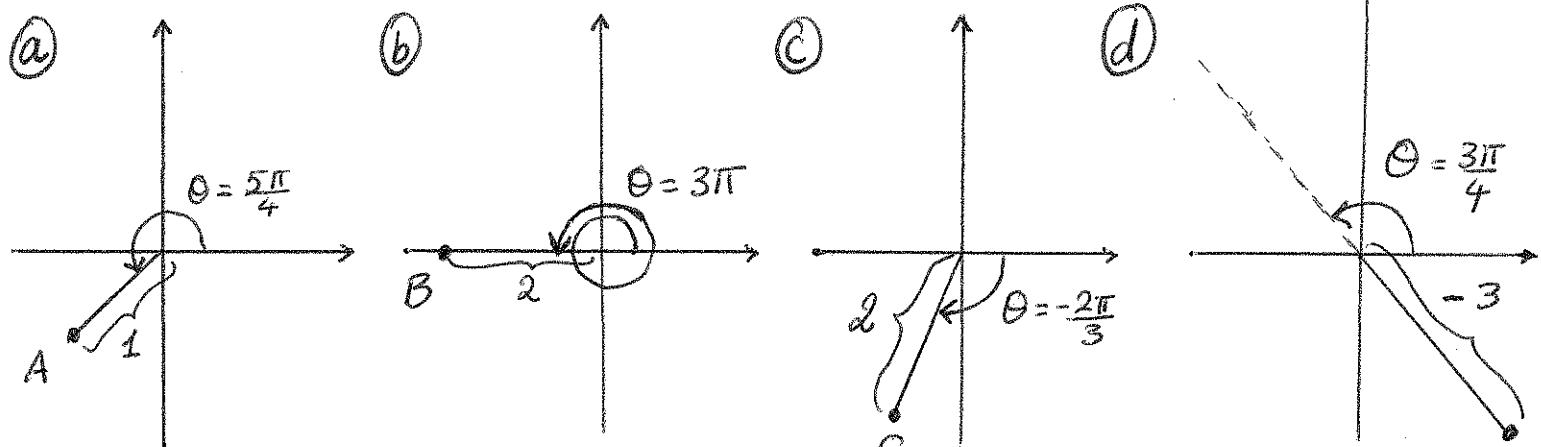
Section 10.3. Polar Coordinates

In Cartesian coordinates, we represent a point in the plane by a pair (x, y) . In polar coordinates, we represent a point by its distance from the origin, " r ", and the angle it makes with the positive x -axis, " θ ". The ordered pair (r, θ) represents a unique point in the plane. We call " r " and " θ " polar coordinates.



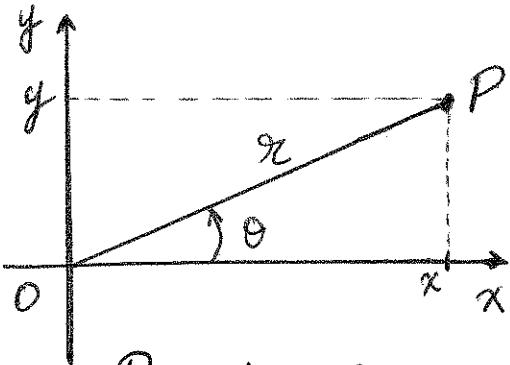
* Can " r " be negative? Yes! The point $(-r, \theta)$ is the reflection of (r, θ) about the origin.

Examples ① Plot the points $A = (1, \frac{5\pi}{4})$, $B = (2, 3\pi)$, $C = (2, -\frac{2\pi}{3})$, and $D = (-3, \frac{3\pi}{4})$, where the coordinates given are polar.



In Cartesian coordinates, every point has one representation (x, y) ; this is not true in polar coordinates: for example $D = (-3, \frac{3\pi}{4})$ is the same as $(3, -\frac{3\pi}{4})$ or $(3, \frac{7\pi}{4})$.

The connection between Cartesian and Polar Coordinates.



observe that $\cos \theta = \frac{x}{r}$, $\sin \theta = \frac{y}{r}$, so

$$x = r \cos \theta, \quad y = r \sin \theta$$

Also,

$$x^2 + y^2 = r^2, \quad \tan \theta = \frac{y}{x}$$

Remarks (1) $r=0$ corresponds to the origin, for ANY value of θ

(2) $\theta=0$ is the (polar) equation of the x -axis.

Ex② Convert the following coordinates from polar to Cartesian: $(2, \frac{\pi}{3})$

$$r=2, \theta=\frac{\pi}{3} \Rightarrow x = 2 \cos \frac{\pi}{3} = 1, \quad y = 2 \sin \frac{\pi}{3} = \sqrt{3}; \text{ Thus } (x,y) = (1, \sqrt{3}).$$

Ex③ Convert the following coordinates from Cartesian to polar: $(1, -1)$.

$$x=1, y=-1 \Rightarrow r = \sqrt{x^2+y^2} = \sqrt{2}, \quad \sin \theta = \frac{y}{r} = -\frac{\sqrt{2}}{2}, \quad \cos \theta = \frac{x}{r} = \frac{\sqrt{2}}{2}. \text{ So } (r, \theta) = (\sqrt{2}, -\frac{\pi}{4}), \text{ or } (\sqrt{2}, \frac{7\pi}{4}).$$

Ex④ what does the polar equation $r=7 \csc \theta$ describe in the Cartesian system? $r = 7 \csc \theta = \frac{7}{\sin \theta} \Rightarrow r \cdot \sin \theta = 7 \Rightarrow y = 7$.

Polar Curves. In Cartesian Coordinates, we looked at functions of the form $y=f(x)$. In polar coordinates, we consider Polar equations of the form $r=f(\theta)$; That is, the distance to the origin is a function of the angle with the x -axis, θ .

Ex⑤ what curve is represented by the polar equation $r=2$?

$r=2$ means any point on the curve in question is at a distance 2 from the origin, no matter what θ is. Thus the curve is the circle

of radius 2 centered at the origin. We can check as follows:

$$r = 2 = \sqrt{x^2 + y^2} \Rightarrow x^2 + y^2 = 4.$$

Ex⑥ what curve is represented by the polar equation $\theta = 1$?

$\theta = 1$ means the angle with the x-axis is fixed at 1 radian, for all values of r, positive and negative. The curve is thus a line through the origin making an angle of 1 radian with the x-axis. We can check as follows: $\tan \theta = \frac{y}{x} = \tan(1) \Rightarrow y = \tan(1) \cdot x$.

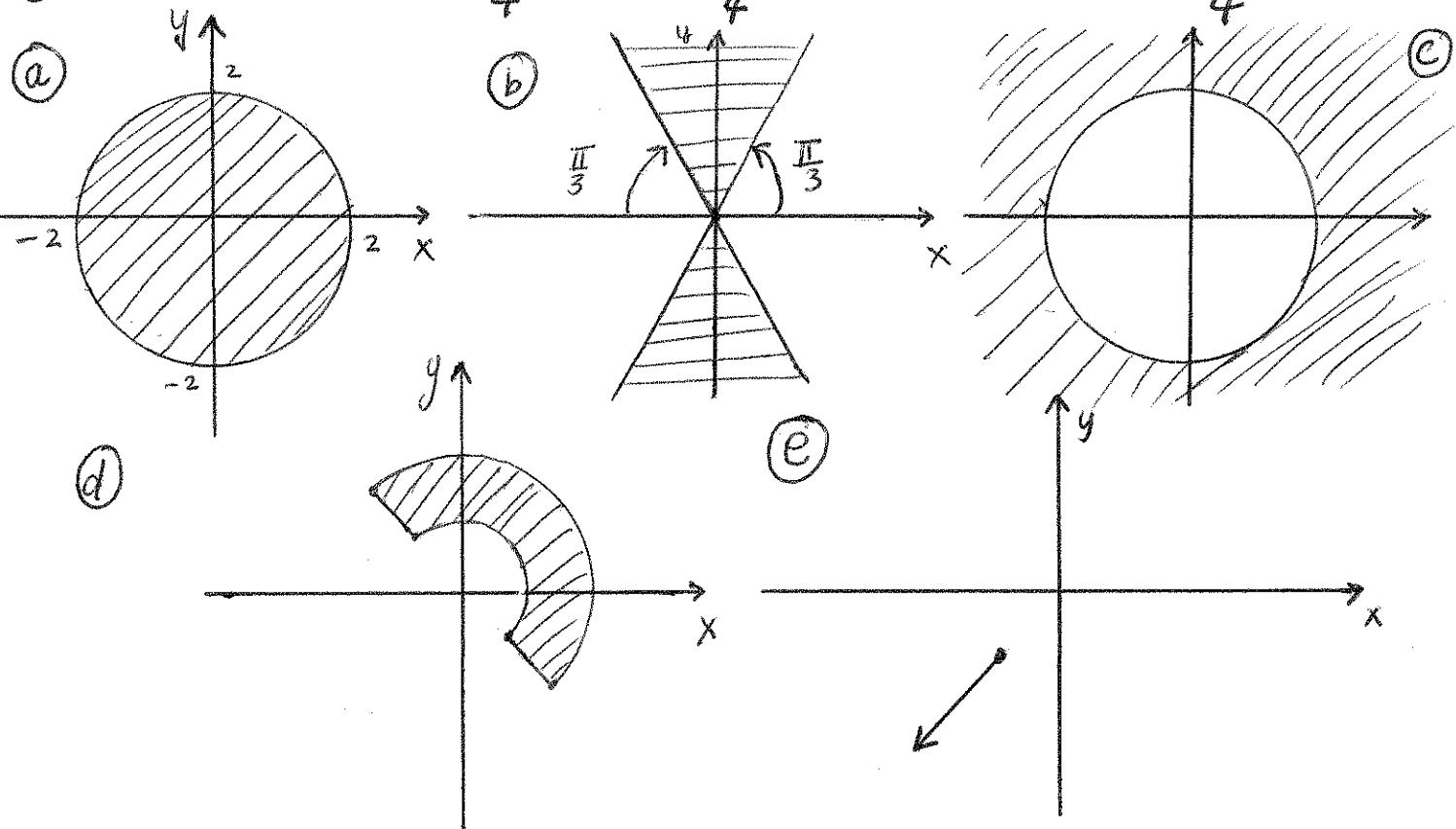
Ex⑦ write the following equation in Cartesian coordinates: $r^2 \sin 2\theta = 9$.

Hint: Use a double angle formula for sin.

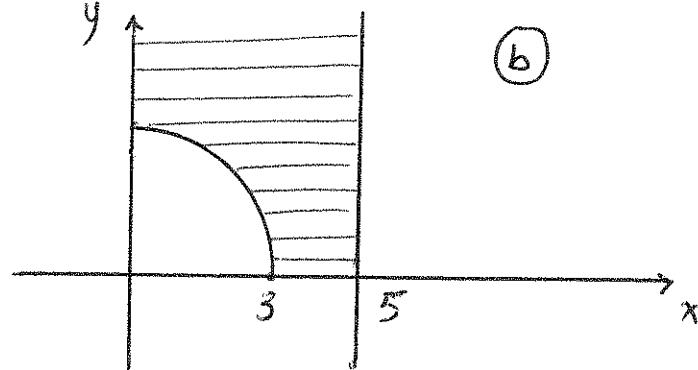
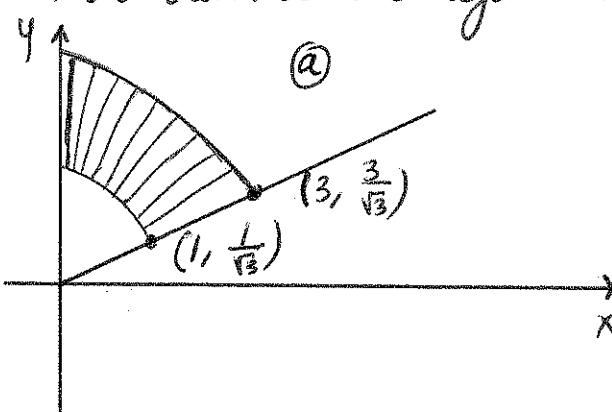
Ex⑧ Sketch the following inequalities

a) $0 \leq r \leq 2$ b) $\frac{\pi}{3} \leq \theta \leq \frac{2\pi}{3}$ c) $x \geq 1$

d) $-2 \leq r \leq -1$, $\frac{3\pi}{4} \leq \theta \leq \frac{7\pi}{4}$ e) $r \geq 1$, $\theta = \frac{5\pi}{4}$.



Ex ⑨ Consider the regions graphed below. Give inequalities for r and θ which describe the regions in polar coordinates.



(a) r ranges from $\sqrt{1^2 + (1/\sqrt{3})^2} = \frac{2}{\sqrt{3}}$ to $\sqrt{3^2 + (3/\sqrt{3})^2} = 2\sqrt{3} \Rightarrow \frac{2}{\sqrt{3}} \leq r \leq 2\sqrt{3}$

θ ranges from $\tan^{-1}\left(\frac{1/\sqrt{3}}{1}\right) = \tan^{-1}\left(\frac{\sqrt{3}}{3}\right) = \frac{\pi}{6}$ to $\theta = \frac{\pi}{2} \Rightarrow \frac{\pi}{6} \leq \theta \leq \frac{\pi}{2}$.

(b) θ ranges from 0 to $\frac{\pi}{2}$; the line $x=5$ has polar equation

$r \cos \theta = 5 \Rightarrow r = 5 \sec \theta$. Thus r ranges from 3 to $5 \sec \theta$:

$$0 \leq \theta \leq \frac{\pi}{2}, \quad 3 \leq r \leq 5 \sec \theta.$$

Tangents to polar Curves. Consider a polar Curve given by $r=f(\theta)$.

Regard θ as a parameter, then

$x=r \cos \theta = f(\theta) \cos \theta, \quad y=r \sin \theta = f(\theta) \sin \theta$: these are parametric equations for x and y in θ .

We know $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\frac{d}{d\theta}(r \sin \theta)}{\frac{d}{d\theta}(r \cos \theta)} = \frac{\frac{dr}{d\theta} \sin \theta + r \cdot \cos \theta}{\frac{dr}{d\theta} \cos \theta - r \cdot \sin \theta}; \quad r=f(\theta)$

Ex ⑩ Find the equation in x and y of the line tangent to the polar Curve $r=f(\theta) = 2 \cos 2\theta$ at $\theta = \frac{\pi}{2}$.

we have $x=r \cos \theta = 2 \cos 2\theta \cos \theta, \quad y=r \sin \theta = 2 \cos 2\theta \sin \theta$.

so, $\frac{dx}{d\theta} = -4 \sin 2\theta \cos \theta - 2 \cos 2\theta \sin \theta, \quad \frac{dy}{d\theta} = -4 \cos 2\theta \sin \theta + 2 \cos 2\theta \cos \theta$

with this, $\frac{dy}{dx} = \frac{-4\sin 2\theta \sin \theta + 2\cos 2\theta \cos \theta}{-4\sin 2\theta \cos \theta - 2\cos 2\theta \sin \theta}$. The slope of the tangent line is $\left. \frac{dy}{dx} \right|_{\theta=\frac{\pi}{2}} = \frac{0}{2} = 0$. The (x, y) point corresponding to $\theta = \frac{\pi}{2}$ is $(x(\frac{\pi}{2}), y(\frac{\pi}{2})) = (2\cos \pi \cdot \cos \frac{\pi}{2}, 2\cos \pi \sin \frac{\pi}{2}) = (0, -2)$. Finally, the equation of the tangent line is $y - (-2) = 0(x - 0)$, or $y = -2$.